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AN ASYMPTOTIC METHOD OF SOLVING TRANSIENT DYNAMIC CONTACT PROBLEMS OF THE THEORY OF ELASTICITY FOR A STRIP†

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An asymptotic method is proposed for solving transient dynamic contact problems of the theory of elasticity for a thin strip. The solution of problems by means of the integral Laplace transformation (with respect to time) and the Fourier transformation (with respect to the longitudinal coordinate) reduces to an integral equation in the form of a convolution of the first kind in the unknown Laplace transform of contact stresses under the punch. The zeroth term of the asymptotic form of the solution of the integral equation for large values of the Laplace parameter is constructed in the form of the superposition of solutions of the corresponding Wiener–Hopf integral equations minus the solution of the corresponding integral equation on the entire axis. In solving the Wiener–Hopf integral equations, the symbol of the kernel of the integral equation in the complex plane is presented in special form – in the form of uniform expansion in terms of exponential functions. The latter enables integral equations of the second kind to be obtained for determining the Laplace–Fourier transform of the required contact stresses, which, in turn, is effectively solved by the method of successive approximations. After Laplace inversion of the zeroth term of the asymptotic form of the solution of the integral equations, the asymptotic solution of the transient dynamic contact problem is determined. By way of example, the asymptotic solution of the problem of the penetration of a plane punch into an elastic strip lying without friction on a rigid base is given. Formulae are derived for the active elastic resistance force on the punch of a medium preventing the penetration of the punch, and the law of penetration of the punch into the elastic strip is obtained, taking into account the elastic stress wave reflected from the strip face opposite the punch and passing underneath it.

1. INTEGRAL EQUATIONS

Transient dynamic contact problems (TDCPs) of the theory of elasticity, concerning the penetration of a rigid punch into an elastic strip using the integral Laplace transformation (with respect to time t) and Fourier transformation (with respect to the longitudinal x coordinate) [1]

$$u^L(x, y, p) = \int_0^\infty u(x, y, t)e^{-pt}dt \tag{1.1}$$

$$u^{LF}(\alpha, y, p) = \int_{-\infty}^\infty u^L(x, y, p)e^{i\alpha x}dx \tag{1.2}$$

successively applied to the differential equations of the theory of elasticity and to the boundary conditions, taking into account the zero initial conditions, reduce to solving an integral equation of the first kind in dimensionless form [2–5]

$$\int_{-1}^{1} \varphi^L(\xi, p)k\left(\frac{\xi - x}{\Lambda}, p\right)d\xi = 2\pi f_0^L(x, p), \quad |x| \le 1 \tag{1.3}$$

$$k(t, p) = \int_\Gamma K(\alpha, p)e^{i\alpha t}d\alpha, \quad f_0^L(x, p) = \mu a^{-1}\Delta f^L(x, p), \quad \Lambda = \frac{c_2}{ap}$$

where $\varphi^L(\xi, p)$ is the Laplace transform of the distribution function of the required contact stresses under the punch, $f^L(x, p)$ is the Laplace transform of the function $f(x, t)$ describing the shape of the punch and the manner in which it penetrates into the elastic medium, μ is the Lamé coefficient of the elastic medium, a is the half-width of the punch, Δ is a certain constant, which depends on the parameter

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$\beta = c_2/c_1$, and c_1 and c_2 are the propagation velocities of the longitudinal and transverse elastic waves of displacements and stresses in the elastic medium. The contour of integration Γ lies in the complex plane $\alpha = \sigma + i\tau$ and passes from $-\infty$ to $+\infty$ at an angle $-\arg p$ to the real axis ($\tau = 0$).

The symbol of the kernel of integral equation (1.3) is the function $K(\alpha, p)$; it is even with respect to α and meromorphic in the complex plane $\alpha = \sigma + i\tau$, where there are two series each of a denumerable set of zeros and poles depending on the parameter p, and, for large and small values of α, has the following asymptotic behaviour

$$K(\alpha, p) = |\alpha|^{-1} + O(|\alpha|^{-3} \exp(-2\gamma\Lambda^{-1}\sigma_2)), \quad \operatorname{Re}(\Lambda^{-1}\sigma_2) > 0 \quad \text{when} \quad \alpha \to \infty \tag{1.4}$$

$$K(\alpha, p) = K(0, p) + O(\alpha^2) \quad \text{when} \quad \alpha \to 0 \tag{1.5}$$

$$\sigma^2 = \sqrt{\alpha^2 + \beta^2}, \quad \gamma = h/a$$

where h is the width of the elastic strip.

In the integral equations of TDCPs of the theory of elasticity, the function $K(\alpha, p)$ is expressed by the ratio of entire functions that are linear combinations of exponential functions

$$\exp(-\gamma\Lambda^{-1}(n\sigma_1 + k\sigma_2)); \quad n + k = 2m; \quad m, n, k = 0, 1, 2 \ldots, \quad \sigma_1 = \sqrt{\alpha^2 + 1}$$

In such a case $K(\alpha, p)$ can be represented by a double functional series

$$K(\alpha, p) = K(\alpha) + K(\alpha)\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} Q_{nk}(\alpha)\exp(-\gamma\Lambda^{-1}(n\sigma_1 + k\sigma_2)) \tag{1.6}$$

$$\operatorname{Re}(\sigma_i) > 0, \quad i = 1, 2$$

where

$$Q_{00}(\alpha) \equiv 0, \quad Q_{nk}(\alpha) = 0; \quad n + k = 2m + 1; \quad m = 0, 1, 2$$

and function $K(\alpha)$ is the symbol of the kernel of the integral equations of TDCPs concerning the penetration of a rigid punch into an elastic half-plane [2, 3].

Series (1.6) converges uniformly in the complex plane $\alpha = \sigma + i\tau$ with sections drawn within it from branching points of the algebraic type $\alpha = \pm i$, $\alpha = \pm i\beta$ to an infinitely distant point $\pm i\infty$. The coefficients of this series $Q_{nk}(\alpha)$ are rational fractional functions of σ_1 and σ_2. When calculating the roots σ_1 and σ_2 in the complex plane, the choice of the branch is determined by condition (1.6), where $\sqrt{1} = 1$.

2. ASYMPTOTIC SOLUTION OF INTEGRAL EQUATION (1.3) FOR LARGE p

After deformation of the contour of integration Γ in the complex plane $\alpha = \sigma + i\tau$ on the real axis ($\tau = 0$), the zeroth term of the asymptotic form of the solution of integral equation (1.3) for small Λ (large p) can be represented in the form of the superposition of functions [6]

$$\varphi^L(x, p) = \varphi_+^L\left(\frac{1+x}{\Lambda}, p\right) + \varphi_-^L\left(\frac{1-x}{\Lambda}, p\right) - \varphi_\infty^L\left(\frac{x}{\Lambda}, p\right) \tag{2.1}$$

which are solutions of the following integral equations

$$\int_{-1}^{\infty} \varphi_+^L(\xi, p) k\left(\frac{\xi - x}{\Lambda}, p\right) d\xi = 2\pi f_0^L(x, p), \quad -1 \leq x < \infty \tag{2.2}$$

$$\int_{-\infty}^{1} \varphi_-^L(\xi, p) k\left(\frac{\xi - x}{\Lambda}, p\right) d\xi = 2\pi f_0^L(x, p), \quad -\infty < x \leq 1 \tag{2.3}$$

$$\int_{-\infty}^{\infty} \varphi_\infty^L(\xi, p) k\left(\frac{\xi - x}{\Lambda}, p\right) d\xi = 2\pi f_0^L(x, p), \quad -\infty < x < \infty \tag{2.4}$$

Integral equations (2.2) and (2.3) as a result of replacement of the variables according to the formulae (a plus sign for (2.2) and a minus sign for (2.3))

$$\pm\xi = \Lambda\xi' - 1, \quad \pm x = \Lambda x' - 1, \quad \pm d\xi = \Lambda d\xi'$$

are transformed into Wiener–Hopf integral equations [6–8] (the primes are omitted)

$$\int_0^\infty \varphi_\pm^L(\xi, p)k(\xi - x, p)d\xi = 2\pi\Lambda^{-1}f_0^L(\pm\Lambda x \mp 1, p), \quad 0 \leqslant x < \infty \tag{2.5}$$

Equation (2.4), after replacement of the variables,

$$\xi = \Lambda\xi', \quad x = \Lambda x', \quad d\xi = \Lambda d\xi'$$

is converted into an integral equation of the convolution type on the entire axis [8]

$$\int_{-\infty}^\infty \varphi_\infty^L(\xi, p)k(\xi - x, p)d\xi = 2\pi\Lambda^{-1}f_0^L(\Lambda x, p), \quad -\infty < x < \infty \tag{2.6}$$

The solution of integral equation (2.6) is obtained by means of integral Fourier transformation (1.1) and is given by the formula

$$\varphi_\infty^L(x, p) = \frac{1}{2\pi\Lambda}\int_{-\infty}^\infty \frac{f_0^{LF}(\alpha, p)}{K(\alpha, p)}e^{-i\alpha x}d\alpha \tag{2.7}$$

Following the general scheme for solving Wiener–Hopf integral equation (2.5) [7, 8], to determine its solution $\varphi_+^L(x, p)$, at the first step, integral equation (2.5) (for the upper signs) is under-defined on the entire real axis, and then, using the integration Fourier transformation (1.2), is reduced to solving the functional equation

$$K(\alpha, p)\varphi_+^{LF}(\alpha, p) = \Lambda^{-1}f_+^{LF}(\alpha, p) + (2\pi)^{-1}e_-^{LF}(\alpha, p) \tag{2.8}$$

which holds in the band $\tau_- < \mathrm{Im}\,(\alpha) < \tau_+$ of the complex plane $\alpha = \sigma + i\tau$. Here, the notation

$$\varphi_+^{LF}(\alpha, p) = \int_0^\infty \varphi_+^L(\xi, p)e^{i\alpha\xi}d\xi, \quad f_{0,+}^{LF}(\alpha, p) = \int_0^\infty f_0^L(\Lambda x - 1, p)e^{i\alpha x}dx$$

$$e_-^{LF}(\alpha, p) = \int_{-\infty}^0 e(x, p)e^{i\alpha x}dx$$

is introduced. The function $\varphi_+^{LF}(\alpha, p)$ is regular in the upper half-plane ($\mathrm{Im}\,(\alpha) > \tau_-$, $-\beta \leqslant \tau_- < 0$, $\beta > 0$), and $e_-^{LF}(\alpha, p)$ is regular in the lower half-plane ($\mathrm{Im}\,(\alpha) < \tau_+$, $0 < \tau_+ \leqslant \beta$, $\beta > 0$).

The next, key stage of the solution of functional equation (2.8) is the factorization of the function $K(\alpha, p)$

$$K(\alpha, p) = K_+(\alpha, p)K_-(\alpha, p)$$

The functions $K_\pm(\alpha, p)$ are regular in the upper ($\mathrm{Im}\,(\alpha) > \tau_-$) and lower ($\mathrm{Im}\,(\alpha) < \tau_+$, $\tau_+ > 0$) half-planes, respectively. In the general case, the determination of $K_\pm(\alpha, p)$ leads to cumbersome singular quadratures [8]. It is more convenient and, as will be shown below, physically more sound to represent the function $K(\alpha, p)$ in functional equation (2.8) in the form of expansion (1.6) in the band $|\mathrm{Im}\,(\alpha)| \leqslant \beta$. Since the functions $K(\alpha)$ [3–5] and $Q_{nk}(\alpha)\exp(-\gamma\Lambda^{-1}(n\sigma_1 + k\sigma_2))$ are regular in the band $|\mathrm{Im}\,(\alpha)| < \beta$, the function $K(\alpha, p)$ in relation (2.8) is regular in this same band of the complex plane $\alpha = \sigma + i\tau$.

Substituting series (1.6) into Eq. (2.8), we obtain the functional equation

$$K(\alpha)\varphi_+^{LF}(\alpha, p) = \Lambda^{-1}f_{0,+}^{LF}(\alpha, p) - K(\alpha)\sum_{n=0}^\infty\sum_{k=0}^\infty M_{nk}(\alpha, p)\varphi_+^{LF}(\alpha, p) + \frac{1}{2\pi}e_-^{LF}(\alpha, p) \tag{2.9}$$

$$M_{nk}(\alpha, p) = Q_{nk}(\alpha)\exp(-\gamma\Lambda^{-1}(n\sigma_1 + k\sigma_2)), \quad n + k = 2m, \quad m = 1, 2, \ldots$$

which is regular in the band $\tau_- < \mathrm{Im}\,(\alpha) < \tau_+$.

Since $K(\alpha)$ is the symbol of the kernel of the integral equation of the corresponding TDCP for the half-plane [2–5], a method was described in [2, 3] for factorization the function $K(\alpha)$, based on a special approximation of $K(\alpha)$, taken in special elementary factorized form. This approximation can be carried out in the complex plane $\alpha = \sigma + i\tau$ with any prescribed accuracy [2].

After factorizing the function $K(\alpha)$ in Eq. (2.9)

$$K(\alpha) = K_+(\alpha)K_-(\alpha) \tag{2.10}$$

followed by division by $K_-(\alpha)$ of the left- and right-hand sides of Eq. (2.9), we obtain the functional equation

$$K_+(\alpha)\varphi_+^{LF}(\alpha, p) = g(\alpha, p) - L(\alpha, p) + \chi_-(\alpha, p) \tag{2.11}$$

where

$$g(\alpha, p) = \frac{f_{0,+}^{LF}(\alpha, p)}{\Lambda K_-(\alpha)}, \quad L(\alpha, p) = K_+(\alpha)\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} M_{nk}(\alpha, p)\varphi_+^{LF}(\alpha, p) \tag{2.12}$$

$$\chi_-(\alpha, p) = \frac{1}{2\pi}\frac{e_-^{LF}(\alpha, p)}{K_-(\alpha)}$$

At the next step of the solution of the problem, we will represent the first two functions of (2.12) in the form of the algebraic sum of two functions, one of which is regular in the upper half-plane, and the second of which is regular in the lower half-plane of the complex plane $\alpha = \sigma + i\tau$

$$g(\alpha, p) = g_+(\alpha, p) + g_-(\alpha, p), \quad L(\alpha, p) = L_+(\alpha, p) + L_-(\alpha, p) \tag{2.13}$$

and which can be defined according to the general theorem [18]

$$g_+(\alpha, p) = \frac{1}{2\pi i}\int_{\Gamma_1}\frac{g(\zeta, p)}{\zeta - \alpha}d\varsigma, \quad g_-(\alpha, p) = g(\alpha, p) - g_+(\alpha, p) \tag{2.14}$$

$$L_+(\alpha, p) = \frac{1}{2\pi i}\int_{\Gamma_2}\frac{L(\zeta, p)}{\zeta - \alpha}d\varsigma, \quad L_-(\alpha, p) = L(\alpha, p) - L_+(\alpha, p) \tag{2.15}$$

The contours of integration Γ_1 and Γ_2 lie in the band of regularity of Eq. (2.9).

Substituting expressions (2.14) and (2.15) into Eq. (2.11), and grouping, in different parts of the inequality, the functions that are regular in the lower and upper half-planes, we obtain a new equality

$$K_+(\alpha)\varphi_+^{LF}(\alpha, p) - g_+(\alpha, p) + L_+(\alpha, p) = g_-(\alpha, p) - L_-(\alpha, p) + \chi_-(\alpha, p) = F(\alpha, p) \tag{2.16}$$

The left-hand side of double equality (2.16) defines the function that is regular in the upper half-plane ($\text{Im}(\alpha) > \tau_-$), and the middle part defines the function that is regular in the lower half-plane ($\text{Im}(\alpha) < \tau_+$). Together, they define a certain function $F(\alpha, p)$ that is regular in the band $\tau_- < \text{Im}(\alpha) < \tau_+$. Assuming that $\varphi_+^{LF}(\alpha, p) \approx \alpha^{1/2}$ ($|\alpha| \to \infty$) and taking into account the fact that the functions $K_\pm(\alpha)$, $g_\pm(\alpha, p)$, and $K_\pm(\alpha, p)$ are decreasing as $|\alpha| \to \infty$, we conclude that, by Liouville's theorem [9], the function $F(\alpha, p)$ under examination in the complex plane is identically equal to zero. In this case, from the double equality (2.16) we obtain two equations for determining the transforms $\varphi_+^{LF}(\alpha, p)$ and $e_-^{LF}(\alpha, p)$

$$\varphi_+^{LF}(\alpha, p) = \frac{g_+(\alpha, p)}{K_+(\alpha)} - \frac{L_+(\alpha, p)}{K_+(\alpha)}, \quad g_-(\alpha, p) - L_-(\alpha, p) + \frac{1}{2\pi}\frac{e_-^{LF}(\alpha, p)}{K_-(\alpha)} = 0 \tag{2.17}$$

Then, taking into account formula (2.15), for determining $\varphi_+^{LF}(\alpha, p)$ we obtain an integral equation of the second kind

$$\varphi_+^{LF}(\alpha, p) = \frac{g_+(\alpha, p)}{K_+(\alpha)} - \frac{1}{2\pi i K_+(\alpha)}\int_{\Gamma}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} M_{nk}(\zeta, p)K_+(\zeta)\frac{\varphi_+^{LF}(\zeta, p)}{\zeta - \alpha}d\zeta \tag{2.18}$$

in which the contour of integration Γ lies in the plane of regularity of functional equation (2.16), and α lies in the upper half-plane, If $\alpha \in \Gamma$, then (2.18) is a singular integral equation of the second kind [8, 10] in which the integral on the right-hand side is understood in the sense of the Cauchy principal part. Introducing the notation

$$\Omega(\alpha, p) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} M_{nk}(\alpha, p), \quad \varphi_{0+}^{LF}(\alpha, p) = K_+(\alpha)\varphi_+^{LF}(\alpha, p) \tag{2.19}$$

we obtain an integral equation of the form

$$\varphi_{0+}^{LF}(\alpha, p) = g_+(\alpha, p) - \frac{1}{2\pi i}\int_\Gamma \Omega(\zeta, p)\frac{\varphi_{0+}^{LF}(\zeta, p)}{\zeta - \alpha}d\zeta \tag{2.20}$$

The series on the right-hand side of the first equality of (2.19) converges uniformly in the complex plane $\alpha = \sigma + i\tau$ with the sections drawn within it that were described in Section 1, and its sum $\Omega(\alpha, p)$ is a function that is continuous in the area of convergence of the series. The following limit holds for it

$$|\Omega(u, p)| \leq Me^{-2\gamma\Lambda^{-1}\beta}, \quad M > 0, \quad u \in \Gamma \tag{2.21}$$

In this case, when $\varphi_{0+}^{LF}(\alpha, p) \in L_q(\Gamma)$ $(1 < q < \infty)$ for a singular operator from the right-hand side of integral equation (2.20)

$$S(\Omega\varphi_{0+}^{LF}) = \frac{1}{2\pi i}\int_\Gamma \Omega(\zeta, p)\frac{\varphi_{0+}^{LF}(\zeta, p)}{\zeta - \alpha}d\zeta \tag{2.22}$$

we will obtain the estimate

$$\| S(\Omega\varphi_{0+}^{LF})\| \leq A_q e^{-2\gamma\Lambda^{-1}\beta} \|\varphi_{0+}^{LF}\|, \quad A_q > 0 \tag{2.23}$$

from which it follows that the operator S is bounded and maps $L_q(\Gamma)$ $(1 < q < \infty)$ into itself [8, 10]; here, $\|\cdot\|$ denotes the norm in the space $L_q(\Gamma)$ $(1 < q < \infty)$.

On the basis of estimate (2.23), it is possible to select γ and Λ such that for large p

$$\|S\|_{L_q} < 1 \tag{2.24}$$

while the solution $\varphi_{0+}^{LF}(\alpha, p)$ from integral equation (2.20) can be obtained by the method of successive approximations [11], and the solution obtained in this way will be unique.

Returning to integral equation (2.20), we form an iteration scheme of the method of successive approximations for solving this integral equation to determine $\varphi_+^{LF}(\alpha, p)$

$$\varphi_{+,m+1}^{LF}(\alpha, p) = \varphi_{+,0}^{LF}(\alpha, p) - \frac{1}{2\pi i K_+(\alpha)}\int_\Gamma \omega(\zeta, p)\frac{\varphi_{+,m}^{LF}(\zeta, p)}{\zeta - \alpha}d\zeta, \quad m = 0, 1, 2, \ldots \tag{2.25}$$

$$\varphi_{+,0}^{LF}(\alpha, p) = \frac{g_+(\alpha, p)}{K_+(\alpha)}, \quad \omega(\zeta, p) = \Omega(\zeta, p)K_+(\zeta)$$

To explain the structure of the solution of integral equation (2.20) or (2.18) obtained by applying the method of successive approximations, we put $m = 0$ in expression (2.25). We obtain

$$\varphi_{+,1}^{LF}(\alpha, p) = \varphi_{+,0}^{LF}(\alpha, p) + \Delta\varphi_{+,0}^{LF}(\alpha, p) \tag{2.26}$$

$$\Delta\varphi_{+,0}^{LF}(\alpha, p) = -\frac{1}{2\pi i K_+(\alpha)}\int_{\Gamma_1} \omega(\zeta_1, p)\frac{\varphi_{+,0}^{LF}(\zeta_1, p)}{\zeta_1 - \alpha}d\zeta_1$$

where $\varphi_{+,0}^{LF}(\alpha, p)$ contains no exponential functions of the form

$$\exp(-\gamma\Lambda^{-1}(n\sigma_1 + k\sigma_2)), \quad n + k = 2m, \quad m, n, k = 0, 1, 2, \ldots \tag{2.27}$$

whereas the integrand in $\Delta\varphi^{LF}_{+,0}(\alpha, p)$ contains an infinite functional series in exponential functions (2.27) $\omega(\zeta_1, p)$ with $n + k = 1, 4, 6, \ldots$. The contour Γ_1 is situated in the band of regularity of Eq. (2.16).

Assuming, now, that $m = 1$, taking expression (2.26) into account we obtain

$$\varphi^{LF}_{+,2}(\alpha, p) = \varphi^{LF}_{+,1}(\alpha, p) + \Delta\varphi^{LF}_{+,1}(\alpha, p) \tag{2.28}$$

$$\Delta\varphi^{LF}_{+,1}(\alpha, p) = \frac{1}{(2\pi i)^2 K_+(\alpha)} \int_{\Gamma_1} \frac{\omega(\zeta_1, p)}{\zeta_1 - \alpha} d\zeta_1 \int_{\Gamma_2} \frac{\omega(\zeta_2, p)}{\zeta_2 - \zeta_1} \Delta\varphi^{LF}_{+,0}(\zeta_2, p) d\zeta_2,$$

where $\varphi^{LF}_{+,1}(\alpha, p)$ contains exponential functions of the form (2.27), whereas $\Delta\varphi^{LF}_{+,1}(\alpha, p)$ also contains an infinite set of such functions due to the product of the double series $\omega(\zeta_1, p)$ and $\omega(\zeta_2, p)$. This means that terms of the series with $\exp(-\gamma\Lambda^{-1}(n\sigma_1 + k\sigma_2))$, with $n + k = 2$, contained in $\varphi^{LF}_{+,1}(\alpha, p)$ will be made no more accurate by the exponential functions contained in $\Delta\varphi^{LF}_{+,1}(\alpha, p)$, since, for these, $n + k = 4, 6, \ldots$. All terms of the series with exponential functions in $\varphi^{LF}_{+,1}(\alpha, p)$ for which $n + k = 4, 6, \ldots$ will be made more accurate by terms of the series with exponential functions contained in $\Delta\varphi^{LF}_{+,1}(\alpha, p)$. The contours Γ_1 and Γ_2 lie in the band of regularity of Eq. (2.16).

Continuing the process of iterations (2.25), we obtain at the mth step

$$\varphi^{LF}_{+,m+1}(\alpha, p) = \varphi^{LF}_{+,m}(\alpha, p) + \Delta\varphi^{LF}_{+,m}(\alpha, p) \tag{2.29}$$

$$\Delta\varphi^{LF}_{+,m}(\alpha, p) = \frac{(-1)^{m+1}}{K_+(\alpha)(2\pi i)^{m+1}} \int_{\Gamma_1} \frac{\omega(\zeta_1, p)}{\zeta_1 - \alpha} d\zeta_1 \int_{\Gamma_2} \frac{\omega(\zeta_2, p)}{\zeta_2 - \zeta_1} d\zeta_2 \ldots \int_{\Gamma_{m+1}} \frac{\omega(\zeta_{m+1}, p)}{\zeta_{m+1} - \zeta_m} \times$$

$$\times \Delta\varphi^{LF}_{+,0}(\zeta_{m+1}, p) d\zeta_{m+1}$$

where $\varphi^{LF}_{+,m}(\alpha, p)$ contains terms of the series with exponential functions of the form (2.27), provided that $n + k = 2, 4, 6, \ldots, 2m, 2m + 2$, and $\Delta\varphi^{LF}_{+,m}(\alpha, p)$ contains an infinite number of terms with exponential functions (2.27) due to the product of the double series $\omega(\zeta_1, p), \omega(\zeta_2, p) \ldots \omega(\zeta_{m+1}, p)$ in an $(m + 1)$-tuple integral, for which $n + k = 2m + 4, 2m + 6, \ldots$. This means that the terms of the series with exponential functions of the form (2.27) with $n + k = 2, 4, 6, \ldots, 2m, 2m + 2$ will be made no more accurate by terms of the series with exponential functions contained in $\Delta\varphi^{LF}_{+,m}(\alpha, p)$, for which $n + k = 2m + 4, 2m + 6, \ldots$. All terms of the series with exponential functions in $\varphi^{LF}_{+,m}(\alpha, p)$, for which $n + k = 2m + 2, 2m + 4, \ldots$, will be made more precise by terms of the series with exponential functions contained in $\Delta\varphi^{LF}_{+,m}(\alpha, p)$. The contours of integration $\Gamma_1, \Gamma_2, \ldots, \Gamma_{m+1}$ lie in the band of regularity of Eq. (2.16).

It follows from the above that the $(m + 1)$th approximation of $\varphi^{LF}_{+,m+1}(\alpha, p)$ to the solution $\varphi^{LF}_{+}(\alpha, p)$ of Eq. (2.16) contains terms with exponential functions (2.27), where $n + k = 2, 4, \ldots, 2m, 2m + 2$, that do not change together with the coefficients of them in subsequent iterations.

This means that, when determining the solution $\varphi^{L}_{+,m+1}(x, p)$ in expansion $K(\alpha, p)$ (1.6), we can confine ourselves to that number of exponential functions for which $n + k \leqslant 2m + 2$, i.e.

$$K(\alpha, p) = K(\alpha) + K(\alpha) \sum_{n=0}^{2m+2} \sum_{k=0}^{2m+2-n} Q_{nk}(\alpha) \exp(-\gamma\Lambda^{-1}(n\sigma_1 + k\sigma_2)) \tag{2.30}$$

$$Q_{00}(\alpha) = 0, \quad Q_{nk}(\alpha) = 0 \quad \text{for} \quad n + k = 2m + 1, \quad m = 0, 1, 2, \ldots$$

Furthermore, it is not difficult to show that the coefficients of such exponential functions in the solution $\varphi^{LF}_{+,m+1}(\alpha, p)$ will be of the same order of accuracy in terms of α as the zeroth term of the solution $\varphi^{LF}_{+,0}(\alpha, p)$. Running ahead, it can be said that, from the physical point of view, the solution $\varphi^{LF}_{+,m+1}(\alpha, p)$, containing terms with exponential functions (2.27) with $n + k = 2, 4, \ldots, 2m, 2m + 2$, is the mathematical description in Laplace–Fourier representations of $m + 1$ repeated reflections of the elastic stress wave generated by the penetration of the punch and reflected $m + 1$ times by the opposite face (with respect to the punch) of the elastic strip, passing under the punch.

The $(m + 1)$th approximation of $\varphi^{LF}_{+,m+1}(\alpha, p)$ of the method of successive approximations of the solution of Eq. (2.18) is taken as the approximate solution $\varphi^{L}_{+}(x, p)$ of integral equation (2.5), after its Fourier inversion

$$\varphi^L_{+,m+1}(x, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi^{LF}_{+,m+1}(\alpha, p) e^{-i\alpha x} d\alpha, \quad m = 0, 1, 2, \ldots \tag{2.31}$$

where $\varphi^{LF}_{+,m+1}(\alpha, p)$ is given by formula (2.29).

The function $\varphi^{LF}_{-,m+1}(\alpha, p)$ is taken as the approximation solution $\varphi^L_{-}(x, p)$ of integral equation (2.5), determined by the same scheme as $\varphi^{LF}_{+,m+1}(\alpha, p)$, during the implementation of which $f_{0,+}(\alpha, p)$ in Eq. (2.8) is defined by the formula

$$f_{0,+}(\alpha, p) = \int_{-\infty}^{\infty} f_0^L(-\Lambda x + 1, p) e^{i\alpha x} dx$$

after its Fourier inversion (2.31), where the plus sign must be replaced by a minus sign.

The zeroth term of the asymptotic solution $\varphi^L(x, p)$ of integral equation (1.3) for large p, containing a description of $m + 1$ repeated reflections of the elastic stress wave passing under the punch, is constructed by means of formula (2.1) with an indication of the number of $m + 1$ repeated reflections using the subscript

$$\varphi^L_{m+1}(x, p) = \varphi^L_{+,m+1}\left(\frac{1+x}{\Lambda}, p\right) + \varphi^L_{-,m+1}\left(\frac{1-x}{\Lambda}, p\right) - \varphi^L_{\infty,m+1}\left(\frac{x}{\Lambda}, p\right) \tag{2.32}$$

where the function $\varphi^L_{\pm,m+1}(x, p)$ is given by formula (2.29), and $\varphi^L_{\infty,m+1}(x, p)$ is given by formula (2.7).

3. THE SOLUTION OF TRANSIENT DYNAMIC CONTACT PROBLEMS

To determine the asymptotic solution of the TDCP for an elastic strip, it is sufficient to apply an inverse Laplace transformation to the solution obtained in the previous section for the integral equation of the TDCP (1.3) of the function $\varphi^L_{m+1}(x, p)$, which is given by formula (2.32). As a result we obtain a function which defines the contact stresses that occur under the punch at $0 < t < 2(m + 2)h/c_1$ in the form

$$\varphi_{m+1}(x, t) = \varphi_{+,m+1}\left(\frac{a(1+x)}{c_2}, t\right) + \varphi_{-,m+1}\left(\frac{a(1-x)}{c_2}, t\right) - \varphi_{\infty,m+1}\left(\frac{ax}{c_2}, t\right) \tag{3.1}$$

The functions on the right-hand side of this expression are the originals of the corresponding functions on the right-hand side of equality (2.32).

4. THE SOLUTION OF A TDCP FOR ELASTIC STRIP

To demonstrate the above method for solving transient dynamic contact problems, we will consider the TDCP of the penetration of a rigid punch of width $2a$ ($|x| \leqslant a, y = 0$) into elastic strip of thickness $h(-\infty < x < \infty, 0 \leqslant y \leqslant h)$ resting on a smooth rigid base ($-\infty < x < \infty, h \leqslant y < \infty$). At the initial instant of time $t = 0$, the rate of penetration of the punch is υ_0, the mass per unit length of the punch is m, and there are no friction forces in the contact zone. The shape of the punch and its low of motion in an elastic medium are determined by the function $f(x, t)$ ($t > 0, x \leqslant \alpha$) (Fig. 1).

At the initial instant of time, the elastic strip is at rest, and therefore the displacements of the elastic medium $u = u(x, y, t)$ and $\upsilon = \upsilon(x, y, t)$ at $t = 0$ and their rates of displacement are taken to be zero. The boundary conditions of such a problem in the generally accepted notation of the theory of elasticity [12] have the form ($t > 0$)

$$\begin{gathered} v(x, 0, t) = f(x, t) \quad (|x| \leqslant a), \quad \sigma_{yy}(x, 0, t) = 0 \quad (a < |x| < \infty) \\ \tau_{xy}(x, 0, t) = v(x, h, t) = \tau_{xy}(x, h, t) = 0 \quad (|x| < \infty) \end{gathered} \tag{4.1}$$

where σ_{yy} and τ_{xy} are the normal and shear stresses.

Using integral Laplace transformation (1.1) and Fourier transformation (1.2), applied successively to the equations of the theory of elasticity in displacements [12] and to mixed boundary conditions (4.1)

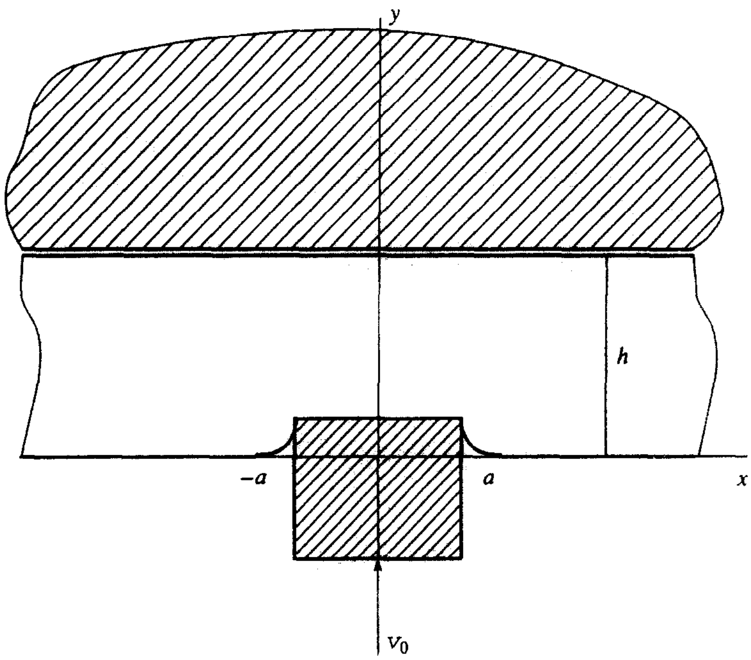

Fig. 1

taking the initial conditions into account, the solution of the TDCP in question reduces to integral equation (1.3), where

$$\begin{aligned}
&\Delta = 2(1-\beta^2)\\
&K(\alpha, p) = 2(1-\beta^2)\sigma_2 R^{-1}(\alpha, p) \qquad (4.2)\\
&R(\alpha, p) = (2\alpha^2+1)^2 \operatorname{ctg}(\gamma\Lambda^{-1}\sigma_2) - 4\alpha^2\sigma_1\sigma_2 \operatorname{cth}(\gamma\Lambda^{-1}\sigma_1)
\end{aligned}$$

The function $K(\alpha, p)$ of the form (4.2) possesses all the properties enumerated in Section 1 and is represented in the form of expansion (1.6), but with the guaranteed presence in expansion $K(\alpha, p)$ of a finite number of terms, the exponents $n + k$ of which should not exceed the prescribed value $2(m + 1)$ $(m = 0, 1, 2, \ldots)$, in the form (2.33), used in implementing the method for solving integral equation (1.3) (here and below, the number $n + k$ will, for convenience, be called the exponential index: $\exp(-\gamma\Lambda^{-1}(n\sigma_1 + k\sigma_2))$).

For the function $K(\alpha, p)$ from (4.2), the coefficients $Q_{nk}(\alpha)$ $(n, k = 0, 1, 2, \ldots)$ of expansion (1.6), (2.30) for even $n + k = 2(m + 1)$ $(m = 0, 1, 2, \ldots)$ are given by the following formulae:

for $n + k = 2$

$$Q_{02}(\alpha) = -2R_1R^{-1}, \quad Q_{20}(\alpha) = 2R_2R^{-1}, \quad Q_{11}(\alpha) = 0 \qquad (4.3)$$

for $n + k = 4$

$$\begin{aligned}
&Q_{04}(\alpha) = 2R_1(R_1+R_2)R^{-2}, \quad Q_{40}(\alpha) = 2R_2(R_1+R_2)R^{-2}\\
&Q_{22}(\alpha) = -8R_1R_2R^{-2}, \quad Q_{31}(\alpha) = Q_{13}(\alpha) = 0
\end{aligned} \qquad (4.4)$$

where

$$R_1 = (2\alpha^2+1)^2, \quad R_2 = 4\alpha^2\sigma_1\sigma_2, \quad R = R_1 - R_2, \quad \sigma_1 = \sqrt{\alpha^2+1}, \quad \sigma_2 = \sqrt{\alpha^2+\beta^2} \qquad (4.5)$$

In this case, in expansion (1.6), (2.30) for $K(\alpha, p)$ from (4.2) $Q_{00}(\alpha) = 0$ and $Q_{nk}(\alpha) = 0$ for $n + k = 2m - 1$ $(m = 1, 2, \ldots)$.

Different methods can be used to obtain the expansion of the function $K(\alpha, p)$ of the form (2.30). We will indicate one of the simplest methods from the technical viewpoint. For simplicity, we will introduce the new notation

$$A = K(\alpha, p), \quad B = K(\alpha), \quad D = g(\alpha, p)$$

The function $K(\alpha, p)$ of the form (4.2) is represented in the form of the relation

$$A = B - DA \tag{4.6}$$

$$B = \Delta\sigma_2 R^{-1}(\alpha), \quad \Delta = 2(1-\beta^2)$$

$$D = R_1 R^{-1}(\operatorname{ctg}\sigma_2^0 - 1) - R_2 R^{-1}(\operatorname{ctg}\sigma_1^0 - 1), \quad \sigma_i^0 = \gamma\Lambda^{-1}\sigma_i, \quad i = 1, 2$$

which can easily be verified, while R_1, R_2, and R are defined by formulae (4.5). Instead of A, we substitute its expression from (4.6) into the right-hand side of equality (4.6), giving

$$A = B - D(B - DA) = B - BD + D^2A$$

Then, substituting, instead of A, its expression from (4.6) into the right-hand side of the latter equality, and continuing such iterations, we obtain at the mth step of the process the following relation

$$A = B - B\sum_{j=1}^{m+1} (-1)^{j+1} D^j + (-1)^{m+2} D^{m+2} A, \quad m = 0, 1, 2, \ldots \tag{4.7}$$

The exponential functions $\exp(-(n\sigma_1^0 + k\sigma_2^0))$ in relation (4.7) are contained only in D and A, and here the final term on the right-hand side of relation (4.7) contains exponential functions whose indices $n + k \geqslant 2(m + 2)$, which is not difficult to show if the formula is used

$$(\operatorname{cth} z - 1)^l = 2^l \sum_{j=l}^{\infty} C_{j-1}^{l-1} \exp(-2jz), \quad \operatorname{Re} z > 0 \tag{4.8}$$

with the expression for D in relation (4.7) raised to the power of $m + 2$. On this basis, the final term indicated must be discarded since, in the expansion of $K(\alpha, p)$ of the form (2.30), it is necessary to retain terms with exponential functions whose indices $n + k \leqslant 2(m + 1)$, especially as, in subsequent iterations, in this case the final term cannot increase the accuracy of the expression given by the terms of the expansion with exponential indices $n + k \leqslant 2(m + 1)$. The product of B and the expression with the summation sign in relation (4.7) contains both terms of the expansion whose exponential indices $n + k \leqslant 2(m + 1)$ and also exponential functions with indices $n + k \geqslant 2(m + 2)$, obtained after using the expression for $(\operatorname{cth} \sigma_i^0 - 1)^l$, according to formula (4.8), in D. Exponential functions with indices $n + k \geqslant 2(m + 2)$ are discarded in the given product in view of the fact that the coefficients of the exponential functions in this product with indices $n + k \geqslant 2(m + 2)$ cannot be made more accurate with m iterations of the process in (4.7) by exponential functions from the last term, the exponential indices of which $n + k \geqslant (2m + 2)$. To make them more accurate, it is necessary to increase the number of iterations m.

Let us consider the process described for representing $K(\alpha, p)$ in the form of expansion (2.30) in more detail.

In the case when $m = 0$, formula (4.7) acquires the form

$$A = B - B\sum_{j=1}^{1} (-1)^{j+1} D^j + D^2 A$$

To retain in the expansion of $K(\alpha, p)$ of the form (2.30) the terms with exponential indices $n + k \leqslant 2$, the final term D^2A is neglected since it contains exponential functions with indices $n + k \geqslant 4$. As a result, the relation

$$A = B - BD$$

is obtained, which, in integral equation notation, has the form

$$K(\alpha, p) = K(\alpha) - K(\alpha)[R_1 R^{-1} S_2 - R_2 R^{-1} S_1], \quad S_i = \operatorname{cth}\sigma_i^0 - 1, \quad i = 1, 2$$

Replacing S according to formula (4.8) when $l = 1$ and retaining in the expansions the exponential functions with indices $n + k \leqslant 2$, i.e. replacing $S_i \Rightarrow 2\exp(-2\sigma_i^0)$, we write the expansion (2.30) for $K(\alpha, p)$ with $n + k \leqslant 2$ in the form of a relation in which the coefficients of the exponential functions are identical with the corresponding coefficients $Q_{nk}(\alpha)$ of the exponential functions in expansion (2.30), represented by formulae (4.3).

In the case $m = 1$, relation (4.7) is represented in the form

$$A = B - B(D^1 - D^2) + D^3 A$$

On the same basis as in the previous case, the last term is neglected and, after changing to integral equation notation, we obtain the expanded form

$$K(\alpha,p) = K(\alpha) - K(\alpha)\left[(R_1R^{-1}S_2 - R_2R^{-1}S_1)^1 - (R_1R^{-1}S_2 - R_2R^{-1}S_1)^2\right]$$

Replacing in the last equation

$$S_i \Rightarrow 2\exp(-2\sigma_i^0) + 2\exp(-4\sigma_j^0),\quad S_i^2 \Rightarrow 4\exp(-4\sigma_i^0)$$

which is required to satisfy the condition $n + k \leqslant 4$, we obtain that $K(\alpha, p)$, when $m = 1$, acquires the form of expansion (2.30), the coefficients of the exponential functions of which are identical with the corresponding $Q_{nk}(\alpha)$ of expansion (2.30) and represented by formulae (4.4) with $n + k = 4$.

Note that, to obtain an expansion of the function $K(\alpha, p)$ of the form (2.30) using a procedure based on the iteration of relation (4.6) with the guaranteed presence in this expansion of terms with exponential indices $n + k \leqslant 2(m + 1)$ ($m = 0, 1, 2, \ldots$), it is necessary, in order to obtain the key relation (4.7), to perform exactly m iterations in relation (4.6).

The formulae obtained above for representing the function $K(\alpha, p)$ (4.2) in the form (2.30) are generalized by the formula

$$K(\alpha,p) = K(\alpha) + K(\alpha)\sum_{j=1}^{m+1}\left[2\sum_{k=1}^{m_*} q_k(\alpha,p)\right]^j,\quad m_* = \left[\frac{m+1}{j}\right],\quad m = 0,1,2,\ldots \tag{4.9}$$

where

$$q_k(\alpha,p) = R_2R^{-1}\exp(-b_k\sigma_1) - R_1R^{-1}\exp(-b_k\sigma_2),\quad \mathrm{Re}(\Lambda^{-1}\sigma_i) > 0,\quad i = 1,2 \tag{4.10}$$
$$b_k = 2k\gamma\Lambda^{-1}$$

which indicates the rule for obtaining the representation of $K(\alpha, p)$ in the form (2.30) for any prescribed number of terms with the guaranteed presence in it of terms with exponential indices $n + k \leqslant 2(m + 1)$ ($m = 0, 1, 2, \ldots$).

We will now write out the solution of integral equation (1.3) of the TDCP considered for the case of a single rereflection of an elastic wave from the upper face and passing under the punch. The symbol of the kernel of integral equation (1.3), i.e. the function $K(\alpha, p)$, in this case is given by formula (4.8). The zeroth term of the asymptotic solution of integral equation (1.3) is written according to formula (2.32) with $m = 0$.

For the case of a plane punch, where $f(x, t) = f(t)$ ($t > 0$), the function $\varphi_{-,1}^L(x, p) = \varphi_{+,1}^L(x, p)$ is determined from formula (2.31) with $m = 0$ as the inverse Fourier transformation of the function $\varphi_{+,1}^{LF}(\alpha, p)$, which is given by formula (2.26) and, after calculating the quadratures contained there, in the case examined acquires the form

$$\varphi_{+,1}^{LF}(\alpha,p) = \varphi_{+,0}^{LF}(\alpha,p) + \sum_{k=1}^{1} 2\varphi_{+,0}^{LF}(\alpha,p)\exp(-b_k\beta) - \frac{1}{\pi}\int_0^\infty L_k(\xi,p)\psi_{+,0}^{LF}(\xi,\alpha,p)d\xi \tag{4.11}$$

$$\varphi_{+,0}^{LF}(\alpha,p) = -\frac{\xi^2-\alpha^2}{2i}\psi_{+,0}^{LF}(\xi,\alpha,p) = -\frac{1}{\Lambda}\frac{f^L(p)}{K_-(0)i\alpha K_+(\alpha)}$$

$$L_k(\xi,p) = q_k(\xi,p) + \exp(-b_k\beta)$$

where $q_k(\xi, p)$ and b_k are given by formulae (4.10), $f^L(p)$ is the Laplace transform of the function $f(t)$, and the summation sign essentially separates symbolically the structure of the new wave under the punch, formed as a result of the arrival under the punch of the elastic stress wave reflected from the opposite face of the strip.

The inverse Fourier transformation of the function $\varphi_{+,1}^{LF}(\alpha, p)$ (4.11) has the form

$$\varphi_{+,1}^{L}(x,p)=\varphi_{+,0}^{L}(x,p)+\sum_{k=1}^{1} 2\varphi_{+,0}^{L}(x,p)\exp(-b_k\beta)-\frac{1}{\pi}\int_0^\infty L_k(\xi,p)\psi_{+,0}^{L}(x,p,\xi)d\xi \tag{4.12}$$

$$\varphi_{+,0}^{L}(x,p)=\theta_0\left[\int_1^\infty \frac{X_1(\xi,x)d\xi}{\xi}+\theta_1\int_\beta^1 \frac{X_2(\xi,x)d\xi}{\xi}+\frac{\pi}{K_+(0)}\right]$$

$$\psi_{+,0}^{L}(x,p,\xi)=-\theta_0\left[\int_1^\infty \frac{X_1(\zeta,x)}{\zeta^2+\xi^2}d\zeta+\theta_1\int_\beta^1 \frac{X_2(\zeta,x)}{\zeta^2+\xi^2}d\zeta\right]$$

$$X_1(\zeta,x)=r(\zeta)\exp\left(\frac{1}{2}d_0(\sqrt{\zeta-\beta}-\sqrt{\zeta-1})^2\right)\exp(-x\zeta)$$

$$X_2(\zeta,x)=r(\zeta)\exp(d_0\zeta)\cos(d_0(\sqrt{\zeta-\beta}\sqrt{1-\zeta})\exp(-x\zeta)$$

$$\theta_0=\frac{1}{\pi\Lambda}\frac{f^L(p)}{K_-(0)},\quad \theta_1=\exp\left(-\frac{1}{2}d_0(1+\beta)\right),\quad r(\zeta)=\frac{\zeta-\eta_0}{\sqrt{\zeta-\beta}}$$

In obtaining formulae (4.11), we used the approximation of the function $K(\alpha)$ from relation (4.6) of the simplest form [2, 3]

$$K(\alpha)=\frac{\sigma_2}{\alpha^2+\eta_0^2}\exp\left[\frac{1}{2}d_0(M_+(\alpha)+M_-(\alpha))\right],\quad M_\pm(\alpha)=(\sqrt{\beta\pm i\alpha}-\sqrt{1\pm i\alpha})^2$$

in which the constant d_0 is determined from the condition for this approximation and the symbol of the kernel $K(\alpha)$ to be identical when $\alpha = 0$

$$d_0=(1-\sqrt{\beta})^{-2}\ln[K(0)\eta_0^2\beta^{-1}]$$

while $\pm i\eta_0$ are the Rayleigh poles determined from the equation $R(\alpha) = 0$, and the function $R(\alpha)$ is given by formulae (4.5). Under the conditions of this approximation we have

$$K_\pm(\alpha)=\frac{\sqrt{\beta\mp i\alpha}}{\eta_0\mp i\alpha}\exp\left[\frac{1}{2}d_0\mp M(\alpha)\right]$$

where $K_+(\alpha) = K_-(-\alpha)$. Other properties of $K_\pm(\alpha)$ were indicated earlier in [2, 3].

The function $\varphi_{\infty,1}^{L}(x,p)$ is the solution of integral equation (2.6), in which terms describing a single rereflection of an elastic wave from the upper face of the strip are retained, and, in the case examined of a plane punch, it is given by a formula obtained from expression (2.7)

$$\varphi_{\infty,1}^{L}(x,p)=\pi\theta_0\left(1+2\sum_{k=1}^{1}\exp(-b_k\beta)\right) \tag{4.13}$$

This formula can be obtained from the solution of integral equation (2.6) by the method of successive approximations using a scheme similar to that described when solving integral equation (2.5).

Having determined, in this way, the asymptotic solution of integral equation (1.3) by formulae (2.32) with $m = 0$, (4.12), and (4.13), to find the solution of the TDCP considered it is sufficient to take the inverse Laplace transformation of function (2.32) with $m = 0$, taking into account expressions (4.12) and (4.13), which is written in the form

$$\varphi(x,t)=\varphi_{+,1}\left(\frac{a(1+x)}{c_2},t\right)+\varphi_{-,1}\left(\frac{a(1-x)}{c_2},t\right)-\varphi_\infty\left(\frac{ax}{c_2},t\right),\quad |x|\leqslant 1,\quad 0<t<4h/c_1 \tag{4.14}$$

$$\varphi_{\pm,1}(u,t)=\varphi_{\pm,0}(u,t)H(t)+\sum_{k=1}^{1}[2\varphi_{\pm,0}(u,t-t_{k1})-\psi_{\pm,0}(u,t)]H(t-t_{k1}) \tag{4.15}$$

$$\varphi_{\pm,0}(u,t)=\mu\Delta_0\left[\frac{d}{dt}\int\limits_{t_{x1}}^{t} f_1(u,\tau)f(t-\tau)d\tau+\frac{d}{dt}\int\limits_{t_{x1}}^{t_{x2}} f_2(u,\tau)f(t-\tau)d\tau+\frac{1}{K_+(0)}(f'(t)+f(0))\right]$$

$$f_1(u,t)=m(u,t)\exp\left(\frac{d_0(1-\beta)^2u}{2(\sqrt{t-\beta u}+\sqrt{t-u})^2}\right) \tag{4.16}$$

$$f_2(u,t)=m(u,t)\exp\left(-\frac{1}{2}d_0(1+\beta)+d_0\frac{t}{u}\right)\cos\left(d_0\frac{\sqrt{t-\beta u}\sqrt{u-t}}{u}\right)$$

$$m(u,t)=\frac{t-\eta_0 u}{\pi t\sqrt{u(t-\beta u)}}$$

$$\psi_{\pm,0}(u,t)=\mu\Delta_0\frac{d}{dt}\int\limits_{t_{k1}}^{t}\left[G(u,\tau)+g\left(\frac{\tau-t_{k1}}{u}\right)\right]f(t-\tau)d\tau \tag{4.17}$$

$$G(u,t)=\frac{1}{\pi^2u}\left[H(\xi_{12}^*-1)\int\limits_0^{\xi_{12}}T_{211}(t,\xi)d\xi-H(\xi_{11}^*-1)\int\limits_0^{\xi_{11}}T_{112}(t,\xi)d\xi+\right.$$

$$\left.+(H(\xi_{12}^*-1)-H(\xi_{22}^*-1))\int\limits_{\xi_{22}}^{\xi_{12}}T_{221}(t,\xi)d\xi-(H(\xi_{11}^*-1)-H(\xi_{21}^*-1))\int\limits_{\xi_{21}}^{\xi_{11}}T_{122}(t,\xi)d\xi\right]$$

$$\xi_{ij}=\frac{\sqrt{(t-uc_2/c_i)^2-t_{kj}^2}}{t_{k2}},\quad \xi_{ij}^*=\frac{t-uc_2/c_i}{t_{kj}},\quad i,j=1,2$$

$$T_{ijn}(t,\xi)=\frac{R_i(\xi)}{R(\xi)}F_j(\theta_n(t,\xi),\xi),\quad \theta_n(t,\xi)=\frac{t-\sigma_n t_{k2}}{t_{k2}},\quad i,j,n=1,2$$

$$\sigma_1=\sqrt{\xi^2+1},\quad \sigma_2=\sqrt{\xi^2+\beta^2}$$

$$F_1(\zeta,\xi)=\frac{\zeta g_1(\zeta)}{\zeta^2+\xi^2},\quad F_2(\zeta,\xi)=\frac{\zeta g_2(\zeta)}{\zeta^2+\xi^2}$$

$$2\pi g(\zeta)=H(\zeta-1)g_1(\zeta)+(H(\zeta-\beta)-H(\zeta-1))g_2(\zeta)$$

$$g_1(\zeta)=\frac{r(\zeta)}{\zeta}\exp\left(\frac{1}{2}d_0(\sqrt{\zeta-\beta}\quad\ -1)^2\right)$$

$$g_2(\zeta)=\frac{r(\zeta)}{\zeta}q(\zeta)\cos\left(d_0\sqrt{\zeta}\quad\sqrt{1-\zeta}\right)$$

$$q(\zeta)=\exp\left(-\frac{1}{2}d_0(1+\beta)+d_0\zeta\right)$$

$$\varphi_{\infty,1}(u,t)=\mu\Delta_1\left[f'(t)+f(0)+2\sum_{k=1}^{1}(f'(t-t_{k1})+f(t_{k1}))\right] \tag{4.18}$$

$$\Delta_0=\Delta/(c_2K_-(0)),\quad \Delta_1=\Delta/(c_2K(0)),\quad \Delta=2(1-\beta^2)$$

$$K_-(0)=\sqrt{K(0)},\quad K(0)=2\beta(1-\beta^2),\quad t_{xi}=a(1+x)/c_i,\quad t_{ki}=2kh/c_i,\quad i=1,2$$

The functions $R_1(\xi)$, $R_2(\xi)$, and $R(\xi)$ are given by formulae (4.5), and the function $r(\zeta)$ is given by the last formula of (4.12).

In the simplest case, where the displacement of the punch is specified by the formula $f(t)=f_0H(t)$, where $H(t)$ is the Heaviside function, which corresponds to the instantaneous penetration of a plane punch into the strip, the solution of TDCP (4.14)–(4.18) considered takes the simplest form, where

$$\varphi_{\pm,0}(u,t)=\mu\Delta_0 f_0\left[f_1(u,t)H(t-t_{x1})+f_2(u,t)(H(t-t_{x1})-H(t-t_{x2}))+\frac{\delta(t)}{K_+(0)}\right]$$

$$\psi_{\pm,0}(u,t)=\mu\Delta_0 f_0[G(u,t)+g(u,t-t_{k1})]H(t-t_{k1})$$

$$\varphi_{\infty,1}(x,t)=\mu\Delta_1 f_0\left[\delta(t)+2\sum_{k=1}^{1}\delta(t-t_{k1})\right]$$

while $\varphi_{\pm,1}(x, t)$ is calculated from formula (4.15), and $\delta(t)$ is the Dirac delta function.

Formulae (4.14) to (4.18) enable us to analyse the nature of the stress wave field under the punch, including the instant of arrival ($t = 2h/c_1$) under the punch of the wave front of the first longitudinal stress wave reflected from the upper face of the strip. When $0 < t < 2h/c_1$ (i.e. before the arrival of the first reflected wave), the field of contact stresses under the punch $\sigma_{yy}(x, 0, t) = -\varphi_1(x, t)$ ($|x| \leq a$) is identical with the field of contact stresses for the corresponding TDCP for an elastic half-plane [2, 3]. When $2h/c_1 < t < 4h/c_1$, the longitudinal elastic stress wave that has passed under the punch and been reflected from the upper face of the strip at the instant of time $t = 2h/c_1$ on the punch edges ($x = \pm a$) generates new waves of contact stresses that propagate from the punch edges as from sources with the velocity of the longitudinal wave c_1. At the front of the longitudinal wave propagating from the punch edges, the contact stresses undergo a break with an integrable root type singularity, and they also have a constant (time-independent) root-type feature at the punch edges ($x = \pm a$).

5. THE MAGNITUDE OF THE FORCE ACTING ON THE PUNCH

The force $P(t)$ acting on the punch, when the punch is displaced in the elastic medium as given by $f(t)$, serves as an integral characteristic of the TDCP and is defined by the formula

$$P(t)=a\int_{-1}^{1}\varphi(x,t)dx,\quad t>0 \tag{5.1}$$

where $\varphi(x, t)$ are the contact stresses under the punch. If one elastic wave rereflected from the upper face of the strip is retained in the solution of the TDCP $\varphi(x, t)$, we have

$$P_1(t)=a\int_{-1}^{1}\varphi_1(x,t)dx,\quad 0<t<\frac{4h}{c_1} \tag{5.2}$$

where $\varphi(x, t)$ is given by formula (4.14). The Laplace transform of the function $P_1(t)$ after calculating the quadrature [13] takes the form

$$P_1^L(p)=a\int_{-1}^{1}\varphi_1^L(x,p)dx=\mu\Delta f^L(p)\Big\{2q_1+q_2p+\sum_{k=1}^{1}[(5q_1+q_2p)\exp(-t_{k1}p)+$$

$$+\frac{1}{2}I_2(p)-\frac{\beta^2}{\pi}I_1(p)]\Big\} \tag{5.3}$$

$$I_i(p)=\int_1^\infty M_i(\beta^{2-i}\sqrt{\xi^2-1})\xi\exp(-t_{k1}\xi p)d\xi],\quad M_i(u)=p_*(u)\frac{R_i(u)}{uR(u)},\quad i=1,2$$

$$p_*(u)=\frac{1}{\pi K_-(0)}\left[\int_1^\infty\frac{g_1(\zeta)}{\zeta^2+\xi^2}d\xi+\int_\beta^1\frac{g_2(\zeta)}{\zeta^2+\xi^2}d\xi\right]$$

$$q_1=\int_1^\infty\frac{g_1(\zeta)}{\zeta}d\zeta+\int_\beta^1\frac{g_2(\zeta)}{\zeta}d\zeta,\quad q_2=\frac{2}{K(0)}\frac{a}{c_2}$$

The functions $g_1(\zeta)$ and $g_2(\zeta)$ are given by formulae (4.17).

After applying an inverse Laplace transformation [13], we obtain

$$P_1(t) = \mu\Delta\{2q_1 f(t) + q_2(f'(t) + f(0)) + \sum_{k=1}^{1} [5q_1(t - t_{k1}) + 2q_2(f'(t - t_{k1}) + f(t_{k1})) +$$

$$+J_2(t) - J_1(t)]\}, \quad 0 < t < 4h/c_1 < 2a/c_1 \tag{5.4}$$

$$J_i(t) = \frac{2}{\pi t_{k2}^2} H(t - t_{ki}) \int_{t_{ki}}^{t} M_i\left(\frac{\sqrt{\tau^2 - t_{ki}^2}}{t_{k2}}\right)\tau f(t - \tau)d\tau, \quad i = 1,2$$

The terms outside the summation sign in expression (5.4) correspond to the formula for the force acting on the punch in the corresponding TDCP for an elastic half-plane; this formula can be obtained from the previous results [5] when $0 < t < 2h/c_1$. Under the summation sign there are expressions that correspond to the complement to the force acting on the punch, after the arrival under the punch of the first stress wave reflected from the upper face of the elastic strip when $2h/c_1 < t < 4h/c_1$.

In the simplest case, when $f(t) = f_0 H(t)$, corresponding to the instantaneous penetration of the punch into the elastic medium, we have

$$P_1(t) = \mu\Delta f_0\left\{2q_i H(t) + q_2\delta(t) + \sum_{k=1}^{1} [5q_1 H(t - t_{k1}) + 2q_2\delta(t - t_{k1}) + J_{2^*}(t) - J_{1^*}(t)]\right\}$$

$$J_{i^*} = J_i \mid f(t) \equiv 1, \quad i = 1,2$$

6. THE MOTION OF A PUNCH IN AN ELASTIC MEDIUM

To establish the law of motion of a punch in an elastic medium $f(t)$, it is assumed that the mass per unit length of the punch m and its velocity at the initial instant $\dot{f}(0) = \upsilon_0$ are known. In this case, the differential equation of motion of a rigid punch as a point mass and the initial conditions have the form

$$m\ddot{f}(t) = Q(t), \quad t > 0; \quad \dot{f}(0) = \upsilon_0, \quad f(0) = f_0 \tag{6.1}$$

where f_0 is the initial displacement of the punch into the elastic medium before the instant of time $t = 0$. The force of elastic resistance of the medium $Q(t) = -P(t)$, governed by the contact stresses between the punch and the elastic strip, is equal to $-P_1(t)$ for the time interval $0 < t < 4h/c_1$ and is determined by formula (5.4).

Using methods of the operational calculus [14] when solving the Cauchy problem (6.1) to determine the Laplace transform $f^L(p)$ of the function $f(t)$, we obtain

$$f^L(p) = \frac{mpf_0 + m\upsilon_0}{mp^2 + w^L(p)}, \quad f^L(p)w^L(p) = P_1^L(p) \tag{6.2}$$

The function P_1^L is given by formula (5.3).

Changing in (6.2) to the asymptotic expression for $f^L(p)$ for large p, and then to the Laplace originals, we obtain the approximate expression

$$f(t) = \upsilon_0 E_1(t) - \upsilon_0 \sum_{k=1}^{1} [u_*(t - t_{k1})E_1(t - t_{k1}) + (\upsilon_* - 2u_*^2)E_2(t) +$$

$$+ \int_{t_{k1}}^{t} \left(w_2(\tau) - \frac{1}{\beta} w_1(\tau)\right) E_2(t - \tau)d\tau], \quad 0 < t < \frac{4h}{c_1} \tag{6.3}$$

$$E_1(t) = H(t)e^{-u_* t}\begin{cases} \sin\omega t/\omega, & \delta_* > 0 \\ \operatorname{sh}\omega t/\omega, & \delta_* < 0 \\ t, & \delta_* = 0 \end{cases}$$

$$E_2(t) = H(t)e^{-u_* t}\begin{cases} (\sin\omega t - \omega t\cos\omega t)/(2\omega^3), & \delta_* > 0 \\ (\operatorname{sh}\omega t - \omega t\operatorname{ch}\omega t)/(2\omega^3), & \delta_* < 0 \\ t^3/6, & \delta_* = 0 \end{cases}$$

$$w_i(t) = \frac{\mu t}{\pi t_{k2}^2} H(t - t_{k2}) M_i\left(\frac{\sqrt{t^2 - t_{ki}^2}}{t_{k2}}\right), \quad i = 1, 2$$

$$u_* = \frac{\mu q_2}{2m}, \quad \delta_* = \frac{2\mu q_1}{m} - \left(\frac{\mu q_2}{2m}\right)^2, \quad v_* = \frac{5\mu q_1}{2m}, \quad \omega = \sqrt{|\delta_*|}$$

Formulae (6.3) for $f(t)$ indicate that the law of motion of the punch will be a decaying motion when $\delta_* \leqslant 0$ and oscillatory decaying when $\delta_* > 0$.

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REFERENCES

1. BETEMAN, H. and ERDELYI, A., *Tables of Integral Transforms*. McGraw-Hill, New York, 1954.
2. ZELENTSOV, V. B., An asymptotic method for solving transient dynamic contact problems. *Prikl. Mat. Mekh.*, 1999, **63**, 2, 303–311.
3. ZELENTSOV, V. B., The transient dynamic contact problem of the penetration of a rigid punch into an elastic half-plane. *Izv. Ross. Akad. Nauk MTT*, 1999, 3, 34–44.
4. PORUCHIKOV, V. B., *Methods of the Dynamic Theory of Elasticity*. Nauka, Moscow, 1986.
5. KOVALENKO, Ye. V. and ZELENTSOV, V. B., Asymptotic methods in transient dynamic contact problems for an elastic half-space. *Zh. Prikl. Mekh. Tekh. Fiz.*, 1997, **38**, 1, 111–119.
6. VOROVICH, I. I., ALEKSANDROV, V. M. and BABESHKO, V. A., *Non-classical Mixed Problems of the Theory of Elasticity*. Nauka, Moscow, 1974.
7. NOBLE, B., *Methods Based on the Wiener–Hopf Technique for the Solution of Partial Differential Equations*. Pergamon Press, London, 1958.
8. ZABREIKO, P. P., KOSHELEV, A. I., KRASNOSEL'SKII, M. A., *et al., Integral Equations: A Reference Text*. Noordhoff Int. Publ., Leyden, 1975.
9. TITCHMARCH, E. C., *The Theory of Functions*. Oxford University Press, London, 1947.
10. PRÖSSDORF, S., *Einige Klassen Singulärer Gleichungen*. Akademie-Verlag, Berlin, 1974.
11. MIKHLIN, S. G., *Linear Integral Equations*. Hindustan Publ. Corp., Delhi, 1960.
12. PARTON, V. Z. and PERLIN, P. I., *Methods of the Mathematical Theory of Elasticity*. Nauka, Moscow, 1981.
13. PRUDNIKOV, A. P., BRYCHKOV, Yu. A. and MARICHEV, O. I., *Integrals and Series*. Vol. 1. Elementary Functions. Gordon and Breach Sci. Publ., New York, 1986.
14. DITKIN, V. A. and PRUDNIKOV, A. P., *Integral Transforms and Operational Calculus*. Pergamon Press, New York, 1965.

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